Math 236W: Section 4.1 Homework

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Proposition 1. *For* $n \geq 4$*,* $0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \ldots + (n-1)n = \frac{n(n-1)(n+1)}{3}$*.*

Collaborators:

Proof. This is proof by induction on n. Consider the base case, $\frac{4*(4-1)*(4+1)}{3} = 20$ and $0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 3 * 4 = 20$. So base case is true. Assume the inductive hypothesis that for some $k \geq 4$ we have that $0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \ldots + (k-1)k = \frac{k(k-1)(k+1)}{3}$. We show that $0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \ldots + (k-1)k + ((k+1)-1)(k+1) = \frac{(k+1)((k+1)-1)((k+1)+1)}{3} = \frac{(k+1)*k*(k+2)}{3} = \frac{k^3+3k^2-2k}{3}$. We have

$$\begin{aligned}
0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \ldots + (k-1)k &= \frac{k(k-1)(k+1)}{3} \\
0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \ldots + (k-1)k + (k+1-1)*(k+1) &= \frac{k(k-1)(k+1)}{3} + (k+1-1)*k+1 \\
&= \frac{k(k-1)(k+1)}{3} + k*(k+1) \\
&= \frac{k*(k-1)*(k+1)}{3} + \frac{3}{3}*(k^2+k) \\
&= \frac{k*(k-1)*(k+1)}{3} + \frac{3*(k^2+3k)}{3}
\end{aligned}$$

$$= \frac{k^3+3k^2+2k}{3} \text{ [Expanding and distributive law].}$$

Hence by induction the result is shown. ☐

Proposition 2. *For all* $n \geq 1$*,* $2 + 4 + 6 + \ldots + 2n = n^2 + n$

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Proof. This is proof by induction on n. Consider the base case $2(1) = 1^2 + 1$. Assume the induction hypothesis that for some $k \geq 1$ we have that $2 + 4 + 6 + \ldots + 2k = k^2 + k$. We show that $2+4+6+\ldots+2*k+2*(k+1) = (k+1)^2+(k+1)$.

We have

$$\begin{aligned}
2+4+6+\ldots+2*k+2*(k+1) &= k^2+k+2*(k+1)\\
&= k^2+k+2k+2\\
&= k^2+3k+2\\
&= k^2+2k+1+(k+1) \text{ [Perfect square]}\\
&= (k+1)^2+(k+1)\\
&= (k+1)^2+(k+1)
\end{aligned}$$

Which is the desired result. ☐

Proposition 3. *For every* $n = 1, 2, 3, \ldots,$ $2^1 + 2^2 + 2^3 + \ldots 2^n = 2^{n+1} - 2$

Collaborators:

Proof. This is proof by induction on n. Consider base case for $n = 1$. $2^1 = 2^2 - 2 = 2$. Assume inductive hypothesis that for $n = k$, $2^1+2^2+2^3+\ldots 2^k = 2^{k+1}-2$. We show that $2^1 + 2^2 + 2^3 + \ldots 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$

$$\begin{aligned}
2^1+2^2+2^3+\ldots 2^k+2^{k+1} &= 2^{k+1}-2+2^{k+1}\\
&= 2^1*2^{k+1}-2\\
&= 2^{k+2}-2
\end{aligned}$$

Which is the desired result. ☐

Proposition 4. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a function such that* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in \mathbb{R}$. *For* $n = 0, 1, 2, \ldots,$ $f(n) = n * f(1)$

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Proof. This is proof by induction on n. We show base case for $n = 0$. See that

$$\begin{aligned}
f(0) &= f(0+0)\\
&= f(0)+f(0)\\
0 &= f(0)
\end{aligned}$$

So base case is true. Assume the inductive hypothesis that $f(k) = k * f(1)$. We show that $f(k+1) = (k+1) * f(1)$.

$$\begin{aligned}
f(k+1) &= f(k)+f(1)\\
&= k*f(1)+f(1)\\
&= f(1)*(k+1)
\end{aligned}$$

Thus the desired result is shown. ☐

Proposition 5. *If $n \in \mathbb{N}$, then $n < 2^n$.*

Collaborators:

Proof. This is proof by induction on n. We show base case for $n = 0$. See that $0 < 2^0$. So base case is true. Assume the inductive hypothesis that $k \leq 2^k$. We show that $(k+1) \leq 2^{k+1}$. See

$$\begin{aligned} k &\leq 2^k \\ k+1 &\leq 2^k + 1 < 2^k + 2^k = 2 * 2^k = 2^{k+1} \end{aligned}$$

Hence $k + 1 \leq 2^{k+1}$ and thus the result is shown. □

Proposition 6. *For every integer $n \geq 0$, $4 \mid (5^n - 1)$.*

Collaborators:

Proof. This is proof by induction on n. See the base case for $n = 0$. Clearly $4 \mid 0$. Assume the inductive hypothesis that there exists $k \geq 0$ such that $4 \mid (5^k - 1)$. We wish to show that $4 \mid (5^{k+1} - 1)$. Since $4 \mid (5^k - 1)$, $\exists \mathrm{u} \in \mathbb{Z}$ such that $(5^k - 1) = 4 * u$. We have

$$\begin{aligned} (5^k - 1) &= 4 * u \\ 5 * (5^k - 1) &= 5 * (4 * u) \\ 5^{k+1} - 5 &= 20 * u \\ 5^{k+1} - 1 &= 20 * u - 4 \\ 5^{k+1} - 1 &= 4 * (5 * u - 1) \end{aligned}$$

Whence $4 \mid (5^{k+1} - 1)$. By induction, $4 \mid (5^n - 1) \forall n \in \mathbb{N}$. □

Proposition 7. *Let $A_1, A_2, \ldots, A_n$ be sets and recall that $\mathcal{P}(A)$ is the powerset of the set A. Then $\mathcal{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathcal{P}(A_1) \cap \mathcal{P}(A_2) \cap \ldots \cap \mathcal{P}(A_n)$ for all integers $n \geq 2$.*

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Proof. This is proof by induction on n. See the base case that $\mathcal{P}(A_1 \cap A_2) = \mathcal{P}(A_2) \cap \mathcal{P}(A_2)$. We show base case by subset inclusion method. Consider the right hand side, $\mathcal{P}(A_2) \cap \mathcal{P}(A_2)$ and let A be a set such that $A \in \mathcal{P}(A_1) \cap \mathcal{P}(A_2)$. Then by definition of intersection we see that

$$\begin{aligned} &\Leftrightarrow \{\ A \in\ \mathcal{P}(A_1) \wedge A \in\ \mathcal{P}A_2\} \\ &\Leftrightarrow \{\ A \in\ A_1 \wedge A \in\ A_2\} \\ &\Leftrightarrow \{\ A \in\ A_1 \cap A_2\} \\ &\Leftrightarrow \{\ A \in\ \mathcal{P}(A_1) \cap \mathcal{P}(A_2)\}. \end{aligned}$$

So base is case is true. Assume the inductive hypothesis that $\exists k$ s.t. $\mathcal{P}(A_1 \cap A_2 \cap \ldots \cap A_k) = \mathcal{P}(A_1) \cap \mathcal{P}(A_2) \cap \ldots \cap \mathcal{P}(A_k)$. We show that $\mathcal{P}(A_1 \cap A_2 \cap \ldots \cap A_k \cap (A_k + 1) = \mathcal{P}(A_1) \cap \mathcal{P}(A_2) \cap \ldots \cap \mathcal{P}(A_k) \cap \mathcal{P}(A_k + 1)$. We have

$\mathcal{P}(A_1 \cap A_2 \cap \ldots \cap A_k) = \mathcal{P}(A_1) \cap \mathcal{P}(A_2) \cap \ldots \cap \mathcal{P}(A_k)$ by inductive hypothesis
Write $U := A_1 \cap A_2 ... \cap A_k$

Then by the fact shown above in our base case and appealing to our inductive hypothesis; that is the power set of the intersection of two sets is equal to powerset of the first set and the powerset of the second set, then we know that $\mathcal{P}(U \cap A_k + 1) = \mathcal{P}(U) \cap \mathcal{P}(A_k + 1)$. Back Substituting our U, we see that $\mathcal{P}(A_1 \cap A_2 \cap \ldots \cap A_k \cap (A_k + 1) = \mathcal{P}(A_1) \cap \mathcal{P}(A_2) \cap \ldots \cap \mathcal{P}(A_k) \cap \mathcal{P}(A_k + 1)$ and whence the desired result is shown. □

Proposition 8. *If* A *and* $B_1, B_2, \ldots, B_n$ *are any sets, then*

$$(A \times B_1) \cap (A \times B_2) \cap \ldots \cap (A \times B_n) = A \times (B_1 \cap B_2 \cap \ldots \cap B_n)$$

for any positive integer $n \geq 2$*.*

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Proof. This is proof by induction on n. We identify the base case for $n \geq 2$, $(A \times B_1) \cap (A \times B_2) = A \times (B_1 \cap B_2)$. However this was precisely shown in proposition 2.34 of our textbook. So the base case is true. Assume the inductive hypothesis $\exists\ k \geq 2$ such that $(A \times B_1) \cap (A \times B_2) \cap \ldots \cap (A \times B_k) = A \times (B_1 \cap B_2 \cap \ldots \cap B_k)$. We show that $(A \times B_1) \cap (A \times B_2) \cap \ldots \cap (A \times B_k) \cap (A \times B_k + 1) = A \times (B_1 \cap B_2 \cap \ldots \cap B_k \cap B_k + 1)$. Let $U = B_2 \cap B_3 ... B_k \cap B_k + 1$. Now observe that

$$(A \times B_1) \cap (A \times B_2) \cap \ldots \cap (A \times B_k) \cap (A \times B_k + 1) = A \times (B_1 \cap U)$$

Using the base case and the inductive hypothesis the result follows.

□