Math 236W: Section 1.3 Homework

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Proposition 1. *There exists a solution to the equation* $\frac{x+3}{3} = \frac{8}{x-2}$.

Collaborators:

Proof. Consider the equation $\frac{x+3}{3} = \frac{8}{x-2}$. We claim that equation has a solution. That solution is

$$x := 5.$$

See that on the left

$$\begin{aligned}\frac{5+3}{3} &= \frac{8}{x-2} \\ (5+3)*(x-2) &= 8*3 \\ 8x-16 &= 24 \\ 8x &= 40 \\ x &= 5\end{aligned}$$

$$\begin{aligned}\frac{x+3}{3} &= \frac{8}{5-2} \\ 3*(x+3) &= 24 \\ 3x+9 &= 24 \\ 3x &= 15 \\ x &= 5\end{aligned}$$

Thus we have $\frac{x+3}{3} = 5 = \frac{8}{x-2}$ and the solution exists. □

Proposition 2. *For each real number* x*, there exists a unique* y *such that* $(x+1)^3 - x^3 = 3y+1$.

Collaborators:

Proof. We claim that $y = x^2 + x$ for this y is unique and exists. We have

$$\begin{aligned} (x+1)^3 - x^3 &= 3y + 1 \\ x^3 + 3x^2 + 3x + 1 - x^3 &= 3y + 1 \\ 3x^2 + 3x + 1 &= 3y + 1 \\ 3x^2 + 3x &= 3y \\ x^2 + x &= y \end{aligned}$$

Since y satisfies the equation, y exists. To show y is unique, suppose u is any other value. We wish to show that $y = u$. We have

$$\begin{aligned} 3y + 1 &= x^3 + 3x^2 + 3x + 1 - x^3 = 3u + 1 \\ 3y + 1 &= 3u + 1 \\ 3y &= 3u \\ y &= u. \end{aligned}$$

Thus the result is shown. ☐

Proposition 3. *Let b, m be real numbers, $m \neq 0$. Then there exists a unique $x \in \mathbb{R}$ such that $0 = mx + b$.*

Collaborators:

Proof. We claim that $x = \frac{-b}{m}$ satisfies the equation $0 = mx + b$. We have

$$\begin{aligned} 0 &= m * (\frac{-b}{m}) + b \\ &= -b + b \\ &= 0. \end{aligned}$$

Since x satisfies the equation, this particular x exists. To show x is unique, suppose u is any other value. We wish to show that $x = u$. We have

$$\begin{aligned} mx + b &= 0 = mu + b \\ mx &= mu \\ x &= u \end{aligned}$$

Thus the result is shown. ☐

Proposition 4. *Let $n > 1$ be an integer. Then there exists an integer m such that $2 \mid m, 3 \mid m, 4 \mid m, \ldots, n \mid m$. Is m unique?*

Collaborators: Office hours with Professor Scoville

Proof. Let n be an integer for $n > 1$. We wish to show there exists an integer m such that $2 \mid m, 3 \mid m, 4 \mid m, \ldots, n \mid m$. We claim that when $m := n!$, where $n! := n * (n-1) * (n-2) ... 2 * 1$, then m has such property so that $2 \mid m, 3 \mid m, 4 \mid m, \ldots, n \mid m$. Let $m := i$, that is where i is the i^{th} m such that $1 \leq i \leq n$, and we will show that $i \mid n!$. Observe that

$$n! = 1 * 2 * 3 * 4 * 5 * 6 \;\; * i \; ... \; n$$

Since $1 \leq i \leq n$ then

$$n! = i * (1 * 2 * 3 * 4 * 5 * 6 * \;\; (i-1) * (i+1) \; ... \; n)$$

Choose $n := i$ and $m := n!$. By choosing $n!$ to be m, and with definition of what is said to be a divisor, we shown that when a multiple is equal to the divisor it can be divided. And since the divisor is equal to the multiple m must be unique. □

Exercise 1. *Prove or find a counterexample: Let a, b, c be positive integers. If $a \mid bc$, then $a \mid b$ or $a \mid c$.*

Collaborators:

Solution 1. *Let a, b, c be positive integers. if $a \mid bc$ then $a \mid b$ or $a \mid c$. We claim that when $a = 15$, $b = 5$, $c = 3$ is a counter example. Observe that $a \mid bc$*

$$15 \mid 5 * 3 = 15|15$$

that is $15 = 15 * q$. Choose $q = 1$ so $a \mid bc$. Consider $a \mid c$ and $a \mid b$:

$$a \mid c = 15 \nmid 5.$$
$$a \mid b = 15 \nmid 3.$$

That is there does not exist integers u, v such that

$$5 = 15 * v.$$
$$3 = 15 * u.$$

Therefore concluding that

$$5 \neq 15 * v.$$
$$3 \neq 15 * u.$$

Exercise 2. *Prove or find a counterexample: There exists an integer* n *such that for every integer* k, $n \mid k$.

Collaborators:

Solution 2. *Let* k *and* n *be integers such that for every* k, $n \mid k$. *That is there exists some* $q \in \mathbb{Z}$ *so that for any* n, $k = n * q$. *Let* u, v *be integers.*

$$k = u * v$$

Choose $u := k$ *and* $q := k - (k - 1)$ *so* $n \mid k$.

Exercise 3. *Prove or find a counterexample: For all integers* $m > 1$, $m^2 - 4$ *is composite.*

Collaborators:

Solution 3. *Let* $m > 1$ *and an integer. Consider* $m^2 - 4$. *We wish to prove that* $\forall m$, $m^2 - 4$ *is composite. We claim that when* $m = 3$ *there is a counter example. Let* $m = 3$. *Observe that*

$$\begin{aligned} &= (3^2) - 4 \\ &= 9 - 4 \\ &= 5 \end{aligned}$$

When $m = 3$ *we see that* $m^2 - 4 = 5$. *Five is a prime number therefore* $\not\forall m, m$ *is composite.*